

ANALYTIC AND TOPOLOGICAL INDEX MAPS WITH VALUES IN THE K -THEORY OF MAPPING CONES

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ABSTRACT. Index maps taking value in the K -theory of a mapping cone are defined and discussed. The resulting index theorem can be view in analogy with the Freed-Melrose index theorem. The framework of geometric K -homology is used in a fundamental way. In particular, an explicit isomorphism from a geometric model for K -homology with coefficients in a mapping cone, C_ϕ , to $KK(C(X), C_\phi)$ is constructed.

1. INTRODUCTION

The Baum-Douglas or (M, E, f) model for K -homology is a fundamental tool in the study of index theory. Since its introduction in [1], it has been used to study both classical and exotic index theory. In particular, it is useful to construct variants of the Baum-Douglas model which are associated to various index problems; for example, models associated to non-integer valued index maps are of interest. We refer to the Baum-Douglas model and its variants as geometric models.

Before introducing our main results, we briefly review the construction of the Baum-Douglas model. The reader can find further details in any of [1, 2, 3, 6]. Let X denote a finite CW-complex. A cycle in the Baum-Douglas model is a triple, (M, E, f) , where M is a compact $spin^c$ -manifold, E is vector bundle, and $f : M \rightarrow X$ is a continuous map. A $\mathbb{Z}/2$ -graded abelian group, $K_*^{geo}(X)$, is obtained by taking equivalence classes of (isomorphism classes) of cycles; the relevant equivalence relation is generated by disjoint union/direct sum, bordism, and vector bundle modification. Furthermore, each part of such a cycle gives an element in KK -theory. These classes are

- (1) the class of the Dirac operator associated to the $spin^c$ -structure on M , $[D] \in KK^*(C(M), \mathbb{C})$;
- (2) the class associated to the vector bundle, $[[E]] \in KK^0(C(M), C(M))$;
- (3) the class associated to the $*$ -homomorphism obtained from the continuous function, $[f] \in KK^0(C(X), C(M))$;

Combining these classes leads to a natural map to analytic K -homology; more precisely, let

$$\mu : (M, E, f) \mapsto [f] \otimes_{C(M)} [[E]] \otimes_{C(M)} [D]$$

This map is an isomorphism that vastly generalizes the Atiyah-Singer index theorem. In particular, the Atiyah-Singer index theorem for compact $spin^c$ -manifold is encoded in the following commutative diagram:

$$\begin{array}{ccc}
K_*^{geo}(X) & \xrightarrow{\mu} & KK^*(C(X), \mathbb{C}) \\
& \searrow \text{ind}_{top} & \swarrow c^* \\
& \mathbb{Z} &
\end{array}$$

where ind_{top} is the topological index map and c^* is the map on KK -theory obtained from the unital inclusion of the complex numbers into $C(X)$.

The setup for our results is as follows. As above, let X be a finite CW-complex. In addition, let $\phi : B_1 \rightarrow B_2$ be a unital $*$ -homomorphism between unital C^* -algebras and C_ϕ be the mapping cone of ϕ . In [5], a geometric model (via Baum-Douglas type cycles) of $KK^*(C(X), C_\phi)$ was constructed. The starting point for this construction was not only the work of Baum and Douglas [1, 2], but also Higson and Roe [9]. The particular case when ϕ is the unital inclusion of the complex numbers into a Π_1 -factor is relevant for \mathbb{R}/\mathbb{Z} -valued index theory; since $K_0(N) \cong \mathbb{R}$, the map at the level of K -theory is the inclusion of the integers into the reals. Further motivation for the construction of this particular geometric model can be found in the introduction of [5].

Let $K_*(X; \phi)$ denote the group defined using these geometric cycles; precise definitions of the cycles and this group are given in Definitions 2.3 and 2.10. The isomorphism from $K_*(X; \phi)$ to $KK^*(C(X), C_\phi)$ considered in [5] was rather indirect. One of the goals of the current paper is the construction of an explicit isomorphism; this map is analogous to the map μ introduced above. It is constructed via neat embeddings of manifolds with boundary into half-spaces. As such, in the case when X is a point, it can be viewed as analogous to the classical topological index map. We obtain a commutative diagram (see Corollary 3.12)

$$\begin{array}{ccc}
K_*(X; \phi) & \xrightarrow{\mu} & KK^*(C(X), SC_\phi) \\
& \searrow \text{ind}_{top} & \swarrow c^* \\
& K_*(SC_\phi) &
\end{array}$$

where μ now denotes the isomorphism constructed in this paper, c^* is the map on KK -theory induces from the $*$ -homomorphism $c : \mathbb{C} \rightarrow C(X)$, and ind_{top} is the topological index map defined in this paper.

There is also an analytic index map defined under the condition that $K_1(B_1) \cong 0$. This rather restrictive condition is used to ensure that the higher Atiyah-Patodi-Singer index can be defined. It is satisfied in the special case when ϕ is the unital inclusion of the complex number into a Π_1 -factor and for other examples used to model geometric K -homology with coefficients (see [5, Example 5.3]). The equality of the two index maps is the content of Theorem 4.7.

This index theorem is analogous to the Freed-Melrose index theorem [7, 8]. As the reader may recall, the Freed-Melrose index theorem provides a topological formula

for the mod k reduction of the index of the Dirac operator with the Atiyah-Patodi-Singer boundary conditions on a compact $spin^c \mathbb{Z}/k\mathbb{Z}$ -manifold. The reader can find more details on $\mathbb{Z}/k\mathbb{Z}$ -manifolds and this index theorem in [7, 8]. Even though the Atiyah-Patodi-Singer index is *not* a topological invariant, it is possible for manifolds with specific geometry data on the boundary to have a topological invariant related to this index. Theorem 4.7 is also of this form since the analytic index is the image of the higher Atiyah-Patodi-Singer index in the K -theory of the mapping cone.

An informal discussion to further articulate the similarities between these index theorems seems in order. One can informally think of the specific geometric data on the boundary limiting the ‘jumps’ in the value of the index associated to Atiyah-Patodi-Singer boundary problem caused by varying the metric. In the case of the Freed-Melrose index theorem, these ‘jumps’ are multiples of k and hence make no contribution to the mod k reduction of the index. In the case considered here, these ‘jumps’ are in the K -theory of B_1 or somewhat more precisely in the image of the map $\phi_* : K_0(B_1) \rightarrow K_0(B_2)$. As such, by exactness, they make no contribution to the index as an element of $K_1(C_\phi)$. In the case of the inclusion of the complex numbers into a II_1 -factor, the ‘jumps’ are integers while the index takes values in \mathbb{R}/\mathbb{Z} .

The prerequisites for the paper are as follows. We assume the reader is familiar with the Baum-Douglas model for K -homology and the basic properties of Hilbert C^* -module bundles. Details on the former can be found in any of [3, 4, 15, 19]. For the latter, we have followed [16] (in particular, see Section 2). In Section 4, properties of the higher Atiyah-Patodi-Singer index theory (see [14] and references therein for details) are used. A number of the constructions considered here require the framework of KK -theory (see for example, [10]). In particular, we generalize a number of constructions from [4] to our setting.

Throughout the paper, B_1 and B_2 denote unital C^* -algebras, $\phi : B_1 \rightarrow B_2$ a unital $*$ -homomorphism, and X a finite CW-complex. If B is a unital C^* -algebra, then the C^* -algebra of continuous B -valued function on X is denoted by $C(X, B)$ and the Grothendieck group of (isomorphism classes of) finitely generated projective Hilbert B -module bundles over X is denoted by $K^0(X; B)$. It is well-known (for example, [16, Proposition 2.17]) that

$$K^0(X; B) \cong K_0(C(X, B)) \cong K_0(C(X) \otimes B)$$

A number of index theorems are required during the paper. Subscript notation is used to clarify which index is in use. For example, the topological index is denoted by ind_{top} , while the higher Atiyah-Patodi-Singer index is denoted by ind_{APS} . Subscript notation is also used in the case of Dirac type operators to specify which manifold it is acting on and if it is twisted by a vector (or Hilbert C^* -algebra module) bundle.

2. REVIEW OF THE GEOMETRIC MODEL

We review the constructions and main results of [5].

Definition 2.1. Let W be a locally compact space, Z a closed subspace of W , and $\phi : B_1 \rightarrow B_2$ a unital $*$ -homomorphism between unital C^* -algebras. Then

$$C^*(W, Z; \phi) := \{(f, g) \in C_0(W, B_2) \oplus C_0(Z, B_1) \mid f|_Z = \phi \circ g\}$$

As the notation suggests, $C^*(W, Z; \phi)$ is a C^* -algebra; it fits into the following pullback diagram:

$$\begin{array}{ccc} C^*(W, Z; \phi) & \longrightarrow & C(Z, B_1) \\ \downarrow & & \downarrow \phi_* \\ C(W, B_2) & \xrightarrow{|_Z} & C(Z, B_2) \end{array}$$

A prototypical example is the case when W is a manifold with boundary and $Z = \partial W$. In particular, the mapping cone of ϕ (denoted by C_ϕ) is an example; it is obtained by taking $W = [0, 1)$ and $Z = pt$. The K_0 -group of $C^*(W, Z; \phi)$ is denoted by $K^0(W, Z; \phi)$. If $g : W \rightarrow W'$ is a continuous map such that $g(Z) \subseteq Z'$, then we obtain a $*$ -homomorphism, $\tilde{g} : C^*(W', Z'; \phi) \rightarrow C^*(W, Z; \phi)$ and hence a map at the level of K -theory groups. We also have a $K^0(W)$ -module structure on $K^0(W, Z; \phi)$ obtained via

$$g \cdot (f_W, f_Z) := (g \cdot f_W, g|_Z \cdot f_Z)$$

where $g \in C(W)$ and $(f_W, f_Z) \in C^*(W, Z; \phi)$.

Definition 2.2. Cycles with vector bundle data

A cycle (over X with respect to ϕ using bundle data) is given by, $(W, (E_{B_2}, F_{B_1}, \alpha), f)$, where

- (1) W is a smooth, compact $spin^c$ -manifold with boundary;
- (2) E_{B_2} is a smooth finitely generated projective Hilbert B_2 -module bundle over W ;
- (3) F_{B_1} is a smooth finitely generated projective Hilbert B_1 -module bundle over ∂W ;
- (4) $\alpha : E_{B_2}|_{\partial W} \cong \phi_*(F_{B_1}) := F_{B_1} \otimes_\phi B_2$ is an isomorphism of Hilbert B_2 -module bundles;
- (5) $f : W \rightarrow X$ is a continuous map.

Definition 2.3. Cycles with K -theory data

A cycle (over X with respect to ϕ using K -theory data) is a triple, (W, ξ, f) , where:

- (1) W is a smooth, compact $spin^c$ -manifold with boundary;
- (2) $\xi \in K^0(W, \partial W; \phi)$;
- (3) $f : W \rightarrow X$ is a continuous map.

The manifold, W , in a cycle need not be connected. As such, a cycle is called even (resp. odd), if each of its connected components are even (resp. odd) dimensional. We also let $\xi_{\partial W}$ and ξ_W denote the images of ξ under the maps $p_1 : K^0(W, \partial W; \phi) \rightarrow K^0(\partial W; B_1)$ and $p_2 : K^0(W, \partial W; \phi) \rightarrow K^0(W; B_2)$ respectively.

The opposite of a cycle, (W, ξ, f) is the same data only W is given the opposite $spin^c$ -structure. It is denote by $-(W, \xi, f)$. The disjoint union of cycles, (W, ξ, f) and $(\tilde{W}, \tilde{\xi}, \tilde{f})$ is given by the cycle:

$$(W \dot{\cup} \tilde{W}, \xi \dot{\cup} \tilde{\xi}, f \dot{\cup} \tilde{f})$$

Two cycles, (W, ξ, f) and $(\tilde{W}, \tilde{\xi}, \tilde{f})$ are isomorphic if there exists a diffeomorphism, $h : W \rightarrow \tilde{W}$ such that h preserves the $spin^c$ -structure, $h^*(\tilde{\xi}) = \xi$, and $\tilde{f} \circ h = f$. Throughout, a “cycle” more precisely refers to an isomorphism class of a cycle.

Definition 2.4. A regular domain, Y , of a manifold M is a closed submanifold of M such that

- (1) $\text{int}(Y) \neq \emptyset$;
- (2) If $p \in \partial Y$, then there exists coordinate chart of M , $\phi : U \rightarrow \mathbb{R}^n$ centered at p such that $\phi(Y \cap U) = \{x \in \phi(U) \mid x_n \geq 0\}$.

Definition 2.5. A bordism (with respect to X and ϕ) is given by (Z, W, η, g) where

- (1) Z is a compact spin^c -manifold with boundary;
- (2) $W \subseteq \partial Z$ is a regular domain;
- (3) $\eta \in K^0(Z, \partial Z - \text{int}(W); \phi)$;
- (4) $g : W \rightarrow X$ is a continuous map.

Remark 2.6. The “boundary” of a bordism, (Z, W, η, F) , is given by $(W, \eta|_W, g|_W)$. The fact that W is a regular domain of ∂W ensures the boundary is indeed a cycle in $K_*(X; \phi)$. Moreover, if (W, ξ, f) is a boundary in the sense of Definition 2.5, then $(\partial W, \xi_{B_1}, f|_{\partial W})$ is a boundary as a cycle in $K_*(X; B_1)$.

Definition 2.7. Two cycles are bordant if there exists bordism with boundary isomorphic to $(W, \xi, f) \dot{\cup} - (W', \xi', f')$. This relation is denoted by

$$(W, \xi, f) \sim_{\text{bor}} (W', \xi', f')$$

Definition 2.8. Let (W, ξ, f) be a cycle and V a spin^c -vector bundle of even rank over W . Then the vector bundle modification of (W, ξ, f) by V is defined to be:

$$(W^V, \pi^*(\xi) \otimes_{\mathbb{C}} \beta_V, f \circ \pi)$$

where

- (1) $\mathbf{1}$ is the trivial real line bundle over W (i.e., $W \times \mathbb{R}$);
- (2) $W^V = S(V \oplus \mathbf{1})$ (i.e., the sphere bundle of $V \oplus \mathbf{1}$);
- (3) β_V is the “Bott element” in $K^0(W^V)$ (see [15, Section 2.5]);
- (4) $\otimes_{\mathbb{C}}$ denotes the $K^0(W^V)$ -module structure of $K^0(W^V, \partial W^V; \phi)$;
- (5) $\pi : W^V \rightarrow W$ is the bundle projection.

The vector bundle modification of (W, ξ, f) by V is often denoted by $(W, \xi, f)^V$.

Remark 2.9. If (W, ξ, f) is a cycle and V is a spin^c -vector bundle of even rank over W , then $(\partial W, \xi_{B_1}, f|_{\partial W})^V|_{\partial W} = \partial(W, \xi, f)^V$.

Definition 2.10. Let \sim be the equivalence relation generated by bordisms and vector bundle modification (i.e., $(W, \xi, f) \sim (W, \xi, f)^V$, for any even rank spin^c -vector bundle, V , over W). Also let

$$K_*(X; \phi) = \{(W, \xi, f)\} / \sim$$

The grading is given as follows. A cycle (W, ξ, f) is said to be even (resp. odd) if the connected components of W are all even (resp. odd) dimensional. Then, $K_0(X; \phi)$ is even cycles modulo \sim and $K_1(X; \phi)$ is likewise only with odd cycles. Note that the relation \sim preserves this grading.

Proposition 2.11. $K_*(X; \phi)$ with the operation of disjoint union is an abelian group. The unit is given by the trivial (i.e., empty cycle) and the inverse of a cycle is given by take its opposite.

Theorem 2.12. If X is a finite CW-complex, then the following sequence is exact:

$$\begin{array}{ccccc}
K_0(X; B_1) & \xrightarrow{\phi_*} & K_0(X; B_2) & \xrightarrow{r} & K_0(X; \phi) \\
\uparrow \delta & & & & \downarrow \delta \\
K_1(X; \phi) & \xleftarrow{r} & K_1(X; B_2) & \xleftarrow{\phi_*} & K_1(X; B_1)
\end{array}$$

where the maps are defined as follows:

- (1) $\phi_* : K_*(X; B_1) \rightarrow K_*(X; B_2)$ takes a B_1 -cycle (M, F_{B_1}, f) to the B_2 -cycle $(M, \phi_*(F_{B_1}), f)$.
- (2) $r : K_*(X; B_2) \rightarrow K_*(X; \phi)$ takes a cycle (M, E_{B_2}, f) to $(M, (E_{B_2}, \emptyset, \emptyset), f)$.
- (3) $\delta : K_*(X; \phi) \rightarrow K_{*+1}(X; B_1)$ takes a ϕ -cycle $(W, (E_{B_2}, F_{B_1}, \alpha), f)$ to the B_1 -cycle $(\partial W, F_{B_1}, f|_{\partial W})$.

3. AN INDEX MAP VIA THE MAPPING CONE AND IMBEDDINGS

Let $H^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_n \geq 0\}$. The next lemma is a consequence of Bott periodicity and the definitions of the objects involved; its proof is left to the reader.

Lemma 3.1. *Let $k \in \mathbb{N}$ and X a finite CW-complex. Then*

$$\begin{aligned}
KK^0(C(X), C^*(H^{2k}, \mathbb{R}^{2k-1}; \phi)) &\cong KK^0(C(X), SC_\phi) \\
KK^0(C(X), C^*(H^{2k+1}, \mathbb{R}^{2k}; \phi)) &\cong KK^0(C(X), C_\phi)
\end{aligned}$$

In particular, $K^0(H^{2k}, \mathbb{R}^{2k-1}; \phi) \cong K^0(SC_\phi)$ and $K^0(H^{2k+1}, \mathbb{R}^{2k}; \phi) \cong K^0(C_\phi)$.

Definition 3.2. Let W and W' be $spin^c$ -manifolds with boundary with dimensions equal mod two and $i : W \rightarrow W'$ a K -oriented neat embedding. The push-forward map induced by i (denoted $i!$) is given by the composition of the Thom isomorphism and the map given by identifying the normal bundle associated with i with a neighbourhood of W' . Thus, the push-forward of i defines a map

$$i! : K^0(W, \partial W; \phi) \rightarrow K^0(W', \partial W'; \phi)$$

This map has two important properties. Firstly, as a map from cocycles of the form, $(E_{B_2}, F_{B_1}, \alpha)$ (see Definition 2.2 and [5]) to K -theory classes, it is given as follows:

- (1) $i!(E_{B_2}, F_{B_1}, \alpha) \mapsto [((\pi_W)^*(E_{B_2}) \otimes \beta_W, (\pi_{\partial W})^*(F_{B_1}) \otimes \beta_{\partial W}, \tilde{\alpha} \otimes id)]$
- (2) $-[(((\pi_W)^*(E_{B_2}) \otimes \tilde{\beta}_W, (\pi_{\partial W})^*(F_{B_1}) \otimes \tilde{\beta}_{\partial W}, \tilde{\alpha} \otimes id)]$

where

- (1) π_W (resp. $\pi_{\partial W}$) is the projection map from the normal bundle (resp. normal bundle restricted to the boundary) to W (resp. ∂W);
- (2) $[\beta_W] - [\tilde{\beta}_W]$ is the Thom class of a normal bundle of W inside W' and $\beta_{\partial W}$ (resp. $\tilde{\beta}_{\partial W}$) is the restriction of β_W (resp. $\tilde{\beta}_W$) to the boundary. The reader should note that the bundles which form the Thom class are not unique, but the resulting K -theory class (i.e., the image of the map $i!$) is unique;
- (3) $\tilde{\alpha}$ is the isomorphism from $(\pi_W)^*(E_{B_2})|_{\partial W'}$ to $(\pi_{\partial W})^*(F_{B_1}) \otimes_\phi B_2$ given by

$$(w, e) \mapsto (w, \alpha(e))$$

Notice that the range of this map is, in fact, $(\pi_{\partial W})^*(F_{B_1} \otimes_\phi B_2)$. However, this bundle can be identified with $(\pi_{\partial W})^*(F_{B_1}) \otimes_\phi B_2$;

Secondly, the map can be realized via the Kasparov product with an element in $KK^0(C^*(W, \partial W; \phi), C^*(W', \partial W'; \phi))$. The construction of this element is as follows. Let ν_W be a normal bundle for $i(W) \subseteq W'$. Then,

$$(3) \quad i! := ((\beta \otimes_{\mathbb{C}} [\tilde{\pi}]) \otimes_{C_0(\nu_W) \otimes C^*(\nu_W, \partial \nu_W; \phi)} [\iota]) \otimes_{C^*(\nu_W, \partial \nu_W; \phi)} [\theta]$$

where

- (1) $\beta \in KK(\mathbb{C}, C_0(\nu_W))$ is the Thom class. It is defined in [4, Appendix 4]; note that we are using the K -theory class rather than the class in $KK(C(W), C_0(\nu_W))$.
- (2) $[\tilde{\pi}] \in KK(C^*(W, \partial W; \phi), C^*(\nu_W, \partial \nu_W; \phi))$ is the KK -theory class obtained from the $*$ -homomorphism $\tilde{\pi} : C^*(W, \partial W; \phi) \rightarrow C^*(\nu_W, \partial \nu_W; \phi)$ defined via $(f_W, g_W) \mapsto (f_W \circ \pi, g_W \circ \pi|_{\partial \nu_W})$ where $\pi : \nu_W \rightarrow W$ is the bundle projection.
- (3) $[\iota] \in KK(C_0(\nu_W) \otimes C^*(\nu_W, \partial \nu_W; \phi), C^*(\nu_W, \partial \nu_W; \phi))$ is the KK -theory class obtained from the $*$ -homomorphism

$$\iota : C_0(\nu_W) \otimes C^*(\nu_W, \partial \nu_W; \phi) \rightarrow C^*(\nu_W, \partial \nu_W; \phi)$$

defined via $h \otimes (f_{\nu_W}, g_{\partial \nu_W}) \mapsto (h \cdot f_{\nu_W}, h|_{\partial \nu_W} \cdot g_{\partial \nu_W})$; here \cdot denotes pointwise multiplication.

- (4) $[\theta] \in KK(C^*(\nu_W, \partial \nu_W; \phi), C^*(W', \partial W'; \phi))$ is the KK -theory class obtained from the $*$ -homomorphism $\theta : C^*(\nu_W, \partial \nu_W; \phi) \rightarrow C^*(W', \partial W'; \phi)$ given by extension by zero.

The reader familiar with pullbacks for C^* -algebras will notice that the definitions of the $*$ -homomorphisms above (e.g., $\tilde{\pi}$, ι , and θ) are obtained naturally from the fact that the C^* -algebras involved are pullbacks. We will often suppress the algebras over which the Kasparov products are taken and use subscript notation when more than one push-forward map is required. For example, in this notation Equation 3, takes the form

$$i! = (\beta_{\nu_W}) \otimes [\tilde{\pi}_{\nu_W}] \otimes [\iota_{\nu_W}] \otimes [\theta_{\nu_W}]$$

Proposition 3.3. *Let $i : W \hookrightarrow W'$ be a neat embedding. Then, the map $i!$ is given by taking the Kasparov product with the class, $[i!]$. Moreover, $i!$ fits into the following commutative diagram:*

$$\begin{array}{ccccc} \rightarrow & K^0(W, \partial W; \phi) & \rightarrow & K^0(W; B_2) \oplus K^0(\partial W; B_1) & \xrightarrow{r_W} & K^0(\partial W; B_2) & \rightarrow \\ & \downarrow i! & & \downarrow i_W! \oplus i_{\partial W}! & & \downarrow i_{\partial W}! & \\ \rightarrow & K^0(W, \partial W'; \phi) & \rightarrow & K^0(W; B_2) \oplus K^0(W'; B_1) & \xrightarrow{r_{W'}} & K^0(\partial W'; B_2) & \rightarrow \end{array}$$

The horizontal morphisms are given by KK -classes associated to the following $*$ -homomorphisms:

- (1) $C^*(W, \partial W; \phi) \rightarrow C_0(W, B_2)$ defined via $(f, g) \mapsto f$;
- (2) $C^*(W, \partial W; \phi) \rightarrow C_0(W, B_2)$ defined via $(f, g) \mapsto g$;
- (3) $C_0(W, B_2) \rightarrow C_0(\partial W, B_2)$ defined via $f \mapsto f|_{\partial W}$;
- (4) $C_0(\partial W, B_1) \rightarrow C_0(\partial W, B_2)$ defined via $f \mapsto \phi \circ f$;

While the vertical morphisms are given by the standard push-forward classes in KK -theory.

Proof. For the first, let $(E_{B_2}, F_{B_1}, \alpha)$ be a cocycle and let $\Gamma(M; E_A)$ denote the continuous section of E_A where E_A is a (finitely generated projective) Hilbert A -module

bundle over M . In this notation, the Kasparov cycle associated to $(E_{B_2}, F_{B_1}, \alpha)$ is given by $\xi = (\mathcal{E}, \rho, 0)$ where

$$\mathcal{E} = \{(s_W, s_{\partial W}) \in \Gamma(W; E_{B_2}) \oplus \Gamma(\partial W; F_{B_1}) \mid (s_W)|_{\partial W} = \alpha \circ (s_{\partial W} \otimes Id_{B_2})\}$$

and ρ is the unital inclusion of the complex number. The product $\xi \otimes_{C^*(W, \partial W; \phi)} [i!]$ can be explicitly computed and (as the reader can verify) is equal to the Kasparov cycle associated to the $i!(E_{B_2}, F_{B_1}, \alpha)$.

The second of the two statements follows from the action of $i!$ on cocycles of the form, $(E_{B_2}, F_{B_1}, \alpha)$, discussed above (see Equation 2). \square

Our goal is the definition of a map, $\mu : K_*(X; \phi) \rightarrow KK^*(C(X), SC_\phi)$. The even case is considered first. Given a cycle (W, ξ, f) in $K_0(X; \phi)$, there exists (for k sufficiently large) a K -oriented neat embedding, $i : W \rightarrow H^{2k}$ and associated KK -theory element $[i!] \in KK(C^*(W, \partial W; \phi), C^*(H^{2k}, \mathbb{R}^{2k-1}; \phi))$. There are also KK -elements associated to ξ and $f : W \rightarrow X$; namely

- (1) $[[\xi]] := \xi \otimes [\iota] \in KK(C(W), C^*(W, \partial W; \phi))$. The reader should recall that $[\iota]$ is defined above; it is the KK -theory class obtained from the $*$ -homomorphism

$$\iota : C_0(\nu_W) \otimes C^*(\nu_W, \partial \nu_W; \phi) \rightarrow C^*(\nu_W, \partial \nu_W; \phi)$$

defined via $h \otimes (f_{\nu_W}, g_{\partial \nu_W}) \mapsto (h \cdot f_{\nu_W}, h|_{\partial \nu_W} \cdot g_{\partial \nu_W})$;

- (2) $[f] \in KK(C(X), C(W))$ is the KK -element naturally associated to the $*$ -homomorphism $\tilde{f} : C(X) \rightarrow C(W)$ induced from f (i.e., $\tilde{f}(g) := f \circ g$);

Combining these three KK -theory elements gives the desired map. More precisely, we have the following definition.

Definition 3.4. Let $\mu : K_0(X; \phi) \rightarrow KK^0(C(X), SC_\phi)$ be the map defined at the level of cycles via

$$\mu(W, \xi, f) := [f] \otimes_{C(W)} [[\xi]] \otimes_{C^*(W, \partial W; \phi)} [i!] \otimes_{C^*(H^{2k}, \mathbb{R}^{2k-1}; \phi)} \mathcal{B}$$

where \mathcal{B} denotes the KK -theory class which gives the map

$$KK(C(X), C^*(H^{2k}, \mathbb{R}^{2k-1}; \phi)) \cong KK(C(X), C^*(H^2, \mathbb{R}; \phi)) = KK(C(X), SC_\phi)$$

obtained via Bott periodicity. The map from $K_1(X; \phi)$ to $KK(C(X), C_\phi)$ is defined in a similar way; one uses a neat embedding into H^{2k+1} (for k sufficiently large). Since Bott periodicity is a natural isomorphism, we often omit the map induced from \mathcal{B} .

A proof that the map μ is well-defined is required. It is standard to show that the map is well-defined at the level of cycles (i.e., independent of the choice of embedding, normal bundle, etc). That it respects the equivalence relation used to define $K_*(X; \phi)$ is more involved.

In particular, further notation and three lemmas are required. The first two lemmas are based on [4, Lemmas 3.5 and 3.6] (the proof of the latter is in Appendix B.2 of [4]). As such, the proofs of the lemmas stated here are similar to those for these lemmas. The final lemma concerns the functorial properties of the push-forward. Again, the proofs is similar to the standard case. The fact that the maps are embeddings simplifies the proofs of these lemmas.

Lemma 3.5. *Let (W, ξ, f) be a cycle in $K_*(X; \phi)$ and V an even rank spin^c -vector bundle over W . Also let $s : W \rightarrow S(V \oplus \mathbf{1})$ be the north-pole section of W into $S(V \oplus \mathbf{1})$ (i.e., $s(w) := (z(w), 1) \in S(V \oplus \mathbf{1})$ where z is the zero section). Then,*

$$(W, \xi, f)^V = (S(V \oplus \mathbf{1}), s!(\xi), f \circ \pi)$$

Proof. Denote $S(V \oplus \mathbf{1})$ by Z . The vector bundle, V , gives a normal bundle of $s(W) \subseteq Z$. Therefore,

$$s! = ([F_V] - [F_V^\infty]) \otimes [\tilde{\pi}_V] \otimes [\iota_V] \otimes [\theta_V]$$

where F_V and F_V^∞ are the vector bundle used to define the Thom isomorphism (see [4, Proposition A.10] for details).

The K -theory class associated to the cycle $(W, \xi, f)^V$ is given by

$$\begin{aligned} \pi_Z^*(\xi) \cdot ([F_Z] - [F_Z^\infty]) &= \xi \otimes [\tilde{\pi}_Z] \otimes ([F_Z] - [F_Z^\infty]) \otimes [\iota_Z] \\ &= \xi \otimes ([F_Z] - [F_Z^\infty]) \otimes [\tilde{\pi}_Z] \otimes [\iota_Z] \\ &= \xi \otimes ([F_V] - [F_V^\infty]) \otimes [\varphi] \otimes [\tilde{\pi}_Z] \otimes [\iota_Z] \end{aligned}$$

where $\varphi : C_0(V) \rightarrow C(Z)$ is the natural inclusion. The reader can check that

$$\iota_Z \circ (id \otimes \tilde{\pi}_Z) \circ (\varphi \otimes id) = \theta_V \circ \iota_V \circ (id \otimes \tilde{\pi}_V)$$

as $*$ -homomorphisms from $C_0(V) \otimes C^*(W, \partial W; \phi)$ to $C^*(Z, \partial Z; \phi)$. The equality of these $*$ -homomorphisms implies that

$$[\varphi] \otimes [\tilde{\pi}_Z] \otimes [\iota_Z] = [\tilde{\pi}_V] \otimes [\iota_V] \otimes [\theta_V]$$

This implies the result. \square

Lemma 3.6. *Let W and W' be smooth, compact spin^c -manifolds with boundary, $i : (W, \partial W) \rightarrow (W', \partial W')$ be a neat embedding and $\xi \in K^0(W, \partial W; \phi)$. Then,*

$$(\xi \otimes_{C^*(W, \partial W; \phi)} [i!]) \otimes_{C^*(W', \partial W'; \phi)} [\iota_{W'}] = [i] \otimes_{C(W)} (\xi \otimes_{C^*(W, \partial W; \phi)} [\iota_W]) \otimes_{C^*(W, \partial W; \phi)} [i!]$$

Proof. Let $p : \mathbb{C} \rightarrow C(W)$ denote the $*$ -homomorphism defined via $\lambda \in \mathbb{C} \mapsto \lambda \cdot 1_W$. It follows from the commutivity of the Kasparov product over \mathbb{C} and direct calculation that

$$\xi = [p] \otimes \iota_W(\xi)$$

where $[p] \in KK^0(\mathbb{C}, C(W))$ is the KK -class associated to p .

Thus, $\iota_{W'}(\xi \otimes i!) = \iota_{W'}([p] \otimes \iota_W(\xi) \otimes i!)$. From this fact, it follows that if $\iota_W(\xi) \otimes i! = (E, \rho, T)$, then $\iota_{W'}(\xi \otimes i!) = (E, \rho', T)$ where ρ' is the composition of the inclusion $C(W') \rightarrow C^*(W', \partial W'; \phi)$ and right action of $C^*(W', \partial W'; \phi)$.

The details are as follows. The Hilbert module in the KK -cycle $\iota_{W'}([p] \otimes \iota_W(\xi) \otimes i!)$ is given by

$$(C(W) \otimes C(W')) \otimes_{C(W) \otimes C(W')} (E \otimes_{\mathbb{C}} C(W')) \otimes_{\iota_{W'}} C^*(W', \partial W'; \phi)$$

As the reader can verify, the map defined on elementary tensors via

$$f_W \otimes g_{W'} \otimes e \otimes h_{W'} \otimes a \mapsto f_W \cdot e \cdot (g_{W'} h_{W'} a)$$

gives a Hilbert $C^*(W', \partial W'; \phi)$ -module isomorphism to E . Moreover, the representation of $C(W')$ on E is the composition of the inclusion $C(W') \rightarrow C^*(W', \partial W'; \phi)$ and right action of $C^*(W', \partial W'; \phi)$. The operator T in the original Kasparov cycle for $\iota_W(\xi) \otimes i!$ also respects this Hilbert module isomorphism.

To proceed further, additional notation is required. Given a locally compact space Y and C^* -algebra A , let $C_b(Y; A)$ be the continuous bounded A -valued functions on Y and

$$C_b^*(\nu_W, \partial\nu_W; \phi) := \{(f, g) \in C_b(\nu_W; B) \oplus C_b(\partial\nu_W; A) \mid f|_{\partial\nu_W} = \phi \circ g\}$$

Let $\pi_{\nu_W} : \nu_W \rightarrow W$ denote the projection map and $\rho_0 : C(W) \rightarrow C_b(\nu_W)$ be the $*$ -homomorphism given by $f \mapsto f \circ \pi_{\nu_W}$.

Using the definition of $i!$, the class in KK -theory, $\xi \otimes [\iota_W] \otimes i!$, can be represented by a Kasparov cycle, (E, ρ, T) , with the following properties:

- (1) E is a Hilbert $C^*(\nu_W, \partial\nu_W; \phi)$ -module (since the Hilbert module in the definition of $i!$ is constructed from a Hilbert $C^*(\nu_W, \partial\nu_W; \phi)$ -module and the inclusion $\theta : C^*(\nu_W, \partial\nu_W; \phi) \rightarrow C^*(W', \partial W'; \phi)$);
- (2) T commutes with the action of $\sigma : C_b(\nu_W) \rightarrow \mathcal{L}(E)$ via multipliers of $C_0(\nu_W)$;
- (3) The map $C(W) \rightarrow \mathcal{L}(E)$ is induced from $\psi_0 : C(W) \rightarrow C_b(\nu_W)$.

Let $h : \nu_W \times [0, 1] \rightarrow \nu_W$ be the map defined by $(x, t) \rightarrow tx$. Then

$$\rho_t : C(W') \xrightarrow{\text{restriction}} C_b(\nu_W) \xrightarrow{oh(\cdot, t)} C_b(\nu_W) \xrightarrow{\sigma} \mathcal{L}(E)$$

defines a homotopy from $\psi_0 \circ i \circ \sigma$ to the restriction map $C(N) \rightarrow C_b(\nu_W)$ composed with σ . These three properties imply that (E, ρ_t, T) is a KK -homotopy from $(E, \rho \circ i, T)$ and (E, ρ', T) . \square

Lemma 3.7. *Let $(W, \partial W)$, $(W', \partial W')$ and $(\tilde{W}, \partial \tilde{W})$ be smooth spin^c -manifolds. If $s : (W, \partial W) \rightarrow (W', \partial W')$ and $i : (W', \partial W') \rightarrow (\tilde{W}, \partial \tilde{W})$ are neat embeddings, then*

$$[s!] \otimes_{C^*(W', \partial W')} [i!] = [(i \circ s)!] \in KK(C^*(W, \partial W; \phi), C^*(\tilde{W}, \partial \tilde{W}; \phi))$$

Proof. We leave the proof to the reader. In fact, we will only need a weaker result: If $\xi \in K^0(W, \partial W; \phi)$, then

$$(\xi \otimes \iota_W) \otimes (i \circ s)! = (\xi \otimes \iota_W) \otimes (s! \otimes i!)$$

This equality follows from a short KK -theory computation using the fact that the push-forward is functorial on K -theory and the previous lemma. \square

Proposition 3.8. *Let (W, ξ, f) be a cycle in $K_*(X; \phi)$ and V a spin^c -vector bundle over W with even dimensional fibers. Then*

$$\mu((W, \xi, f)^V) = \mu(W, \xi, f) \text{ in } KK^*(C(X), SC_\phi)$$

Proof. Let $Z = S(V \oplus \mathbf{1})$, $i_Z : Z \rightarrow H^n$ be a neat embedding (we take n even for even cycles and n odd for odd cycles), and $s : W \rightarrow Z$ be the neat embedding of W into Z via the north pole section of Z . The definition of μ , the fact that $s \circ \pi_W = id$, and the previous three lemmas imply that

$$\begin{aligned} \mu((W, \xi, f)^V) &= [f] \otimes [\pi_Z] \otimes [[s!(\xi)]] \otimes [i_Z!] \\ &= [f] \otimes [\pi_W] \otimes \iota_Z(s!(\xi)) \otimes [i_Z!] \\ &= [f] \otimes [\pi_W] \otimes [s] \otimes \iota_W(\xi) \otimes [s!] \otimes [i_Z!] \\ &= [f] \otimes [s \circ \pi_W] \otimes \iota_W(\xi) \otimes [(i_Z \circ s)!] \\ &= [f] \otimes [[\xi]] \otimes [(i_Z \circ s)!] \\ &= \mu(W, \xi, f) \end{aligned}$$

The last equality follows since $i_Z \circ s$ is a neat embedding (of W into H^n) and the independence of the definition of μ on the choice of embedding. \square

The bordism relation is considered next, but first some additional notation is introduced. Recall that

$$H^{2k} = \{(x_1, \dots, x_{2k}) \in \mathbb{R}^{2k} \mid x_{2k} \geq 0\}$$

and let

$$H_-^{2k} := \{(x_1, \dots, x_{2k}) \in \mathbb{R}^{2k} \mid x_{2k} \leq 0\}$$

We will make use of the C^* -algebras $C^*(H^{2k}, \mathbb{R}^{2k-1}; \phi)$ and $C^*(\mathbb{R}^{2k}, H_-^{2k}; \phi)$ along with the natural maps

- (1) $R : C^*(\mathbb{R}^{2k}, H_-^{2k}; \phi) \rightarrow C^*(H^{2k}, \mathbb{R}^{2k-1}; \phi)$ defined by restriction;
- (2) $I : C_0(\mathbb{R}^{2k}; B_2) \rightarrow C^*(\mathbb{R}^{2k}, H_-^{2k}; \phi)$ defined via $f \mapsto (\tilde{f}, 0)$ where

$$\tilde{f} = \begin{cases} f(x) & : x \in H^{2k} \\ 0 & : x \in H_-^{2k} \end{cases}$$

(the well-definedness of \tilde{f} follows from the fact that f vanishes at ∞);

- (3) $\tilde{I} : C_0(\mathbb{R}^{2k}; B_2) \rightarrow C^*(H^{2k}, \mathbb{R}^{2k-1}; \phi)$ defined via $(f, g) \mapsto (f, 0)$.

It follows from these definitions that $R \circ I = \tilde{I}$.

Proposition 3.9. *If (W, ξ, f) is a boundary in the sense of Definition 2.6, then $\mu(W, \xi, f)$ is trivial in $KK^*(C(X), SC_\phi)$.*

Proof. We prove the result for even cycles; the odd case is similar. The reader should recall the notation introduced just before the statement of the proposition. Let (W, ξ, f) be a cycle in $K_0(X; \phi)$ which is the boundary of $((Z, W), \eta_Z, g)$. Fix an embedding $j : \partial Z \hookrightarrow \mathbb{R}^{2k}$ such that the restriction of j to $W \subseteq \partial Z$ is a neat embedding of $W \rightarrow H^{2k}$. Denote $j|_W$ by i . Let ν_j be a normal bundle for $j(Z) \subseteq \mathbb{R}^{2k}$. Then $\nu_i := \nu_j|_{H^{2k}}$ is a normal bundle for $i(W) \subseteq H^{2k}$.

By definition, $\mu(W, \xi, f) = [f] \otimes [[\xi]] \otimes [i!] \in KK^0(C(X), C^*(H^{2k}, \mathbb{R}^{2k-1}; \phi))$. Let (M, η, h) denote $(\partial Z, (\eta_W)_{B_2}|_{\partial Z}, g|_{\partial Z})$ and

$$\mu_{B_2}(M, \eta, h) := [h] \otimes_{C(M)} [[\eta]] \otimes_{C(M)} [j!] \in KK^0(C(X), C_0(\mathbb{R}^{2k}) \otimes B_2)$$

Standard results (see for example, [19]) imply that μ_{B_2} is a well-defined map from $K_0(X; B)$ to $KK^0(C(X), B_2)$. In particular, μ_{B_2} vanishes on boundaries. Hence $\mu_{B_2}(M, \eta, h) = 0$ (since (M, η, h) is a boundary in $K_*(X; B_2)$). This observation reduces the proof to showing that

$$(4) \quad \mu(W, \xi, f) = \tilde{I}_*(\mu_{B_2}(M, \eta, h))$$

where $\tilde{I}_* : KK^0(C(X), C_0(\mathbb{R}^{2k}) \otimes B_2) \rightarrow KK^0(C(X), C^*(H^{2k}, \mathbb{R}^{2k-1}; \phi))$ is the map on KK -theory induced from the $*$ -homomorphism, \tilde{I} .

Let $N \in \mathbb{N}$ be sufficiently large so that the normal bundle ν_j translated by $(0, \dots, 0, N)$ is contained in $\text{int}(H^{2k})$. For $t \in [0, 1]$, let j_t denote the embedding of M into \mathbb{R}^{2k} defined via $j_t(m) := j(m) + (0, \dots, 0, Nt)$. For each t , let

$$\mu_{j_t}(M, \eta, h) = [h] \otimes_{C(M)} [[\tilde{\eta}_t]] \otimes_{C^*(M, W_t; \phi)} [j_t!] \in KK^0(C(X), C^*(\mathbb{R}^{2k}, H^{2k-1}; \phi))$$

where $W_t := j_t(M) \cap H_-^{2k}$ and $[[\tilde{\eta}_t]] \in KK(C(M), C^*(M, W_t; \phi))$ the image of η under the map induced from the $*$ -homomorphism, $C^*(M, W; \phi) \rightarrow C^*(M, W_t; \phi)$ defined via $(f, g) \mapsto (f, g|_{W_t})$. It follows from the definitions of I , R , j_t , etc that

$$R_*(\mu_{j_0}(M, \nu, h)) = \mu(W, \xi, f) \text{ and } \mu_{j_1}(M, \nu, h) = I_*(\mu_{B_2}(M, \nu, h))$$

Moreover, $\mu_{j_t}(M, \nu, h)$ defines a homotopy between the KK -cycles $\mu_{j_0}(M, \nu, h)$ and $\mu_{j_1}(M, \nu, h)$. Hence

$$\begin{aligned} \mu(W, \xi, f) &= R_*(\mu_{j_0}(M, \nu, h)) \\ &\sim R_*(\mu_{j_1}(M, \nu, h)) \\ &= (R \circ I)_*(\mu_{B_2}(M, \nu, h)) \\ &= \tilde{I}_*(\mu_{B_2}(M, \nu, h)) \end{aligned}$$

As noted in Equation 4, this implies the result. \square

Theorem 3.10. *If X is a finite CW-complex, then the map $\mu : K_*(X; \phi) \rightarrow KK^*(C(X), SC_\phi)$ is an isomorphism.*

Proof. The main step is to show that the following diagram commutes:

$$\begin{array}{ccccccccc} \rightarrow & K_0(X; B_1) & \xrightarrow{\phi_*} & K_0(X; B_2) & \xrightarrow{r} & K_0(X; \phi) & \xrightarrow{\delta} & K_1(X; B_1) & \rightarrow \\ & \mu_{B_1} \downarrow & & \mu_{B_2} \downarrow & & \mu \downarrow & & \mu_{B_1} \downarrow & \\ \rightarrow & KK^0(C(X), B_1) & \xrightarrow{\phi_*} & KK^0(C(X), B_2) & \xrightarrow{r_{ana}} & KK^0(C(X), SC_\phi) & \xrightarrow{\delta_{ana}} & KK^1(C(X), B_1) & \rightarrow \end{array}$$

where

- (1) The first exact sequence is from Theorem 2.12;
- (2) The vertical maps, μ_{B_i} ($i = 1, 2$), are defined at the level of cycles via $\mu_{B_i}(M, E_{B_i}, f) = [f] \otimes_{C(M)} [[E_{B_i}]] \otimes_{C(M)} [D_M]$ (see [19] for details);
- (3) The second exact sequence is the long exact sequence in KK -theory obtained from the short exact sequence of C^* -algebras

$$0 \rightarrow SB_2 \rightarrow C_\phi \rightarrow B_1 \rightarrow 0$$

Again, the details of commutativity are given in the case of even cycles; the odd case is similar. That $\mu_{B_2} \circ \phi_* = \phi_* \circ \mu_{B_1}$ is standard. With the goal of showing that $r_{ana} \circ \mu_{B_2} = \mu \circ r_{geo}$ in mind, let (M, E_{B_2}, f) be a geometric cycle in $K_0(X; B_2)$. Then

$$(r_{ana} \circ \mu_{B_2})(M, E_{B_2}, f) = r_{ana}([f] \otimes [[E_{B_2}]] \otimes [i!])$$

where $i : M \rightarrow \mathbb{R}^{2k}$ is an embedding. But r_{ana} is given by the inclusion of $C_0(\mathbb{R}^{2k}) \otimes B_2 \rightarrow C^*(H^{2k}, \mathbb{R}^{2k-1}; \phi)$. It is induced from the natural inclusion, $\hat{r} : \mathbb{R}^{2k} \hookrightarrow H^{2k}$. However, the map $i \circ \hat{r}$ is a (neat) embedding of $M \rightarrow H^{2k}$. Using this embedding in the definition of μ , leads to the result.

Next, the proof that $\mu_{B_1} \circ \delta_{geo} = \delta_{ana} \circ \mu$ is considered. Let (W, ξ, f) be a cycle in $K_0(X; \phi)$ and $i : W \hookrightarrow H^{2k}$ a neat embedding. Then

$$\mu_{B_1}(\delta_{geo}(W, \xi, f)) = \mu_{B_1}(\partial W, \xi_{B_1}, f|_{\partial W}) = [f|_{\partial W}] \otimes [[\xi_{B_1}]] \otimes [i|_{\partial W}!]$$

Whereas

$$(\delta_{ana} \circ \mu)(W, \xi, f) = \delta_{ana}([f] \otimes [[\xi]] \otimes [i!]) = [f] \otimes [[\xi]] \otimes [i!] \otimes [ev_{\mathbb{R}^{2k}}]$$

where $ev_{\mathbb{R}^{2k}} : C^*(H^{2k}, \mathbb{R}^{2k-1}; \phi) \rightarrow C_0(\mathbb{R}^{2k-1}) \otimes B_1$ is given by $(f, g) \rightarrow g$. To compare these KK -classes, three $*$ -homomorphisms are required; they are

- (1) $\gamma : C^*(W, \partial W; \phi) \rightarrow C(\partial W) \otimes B_1$ is defined via $(f, g) \mapsto g$;
- (2) $\iota_W : C(W) \otimes C^*(W, \partial W; \phi) \rightarrow C^*(W, \partial W; \phi)$ is defined above in the discussion following Equation 3;
- (3) $r_W : C(W) \rightarrow C(\partial W)$ is the restriction to the boundary (i.e., $r_W(f) = f|_{\partial W}$);

The KK -classes associated to these $*$ -homomorphisms satisfy the following

- (1) $[f|_{\partial W}] = [f] \otimes_{C(W)} [r_W]$;
- (2) $[r_W] \otimes [[\xi_{B_1}]] = [[\xi]] \otimes \gamma$;
- (3) $[i!] \otimes [ev_{\mathbb{R}^{2k}}] = [\gamma] \otimes [i|_{\partial W}!]$.

The proofs of these equalities follows from standard properties of KK -theory. The first equality is standard. In regards to the second (i.e., showing that $[r_W] \otimes [[\xi_{B_1}]] = [[\xi]] \otimes \gamma$), we consider the case when ξ is given by a triple $(E_{B_2}, F_{B_1}, \alpha)$ (rather than a formal difference of such triples). The general case follows easily from this case. If E_A is a (finitely generated projective) Hilbert A -module bundle over M , then let $\Gamma(M; E_A)$ denote the continuous sections of E_A .

Using this notation, the Kasparov cycle $[(E_{B_2}, F_{B_1}, \alpha)]$ is given by $(\mathcal{E}, \rho, 0)$ where

$$\mathcal{E} = \{(s_W, s_{\partial W}) \in \Gamma(W; E_{B_2}) \oplus \Gamma(\partial W; F_{B_1}) \mid (s_W)|_{\partial W} = \gamma \circ (s_{\partial W} \otimes Id_{B_2})\}$$

and, for $g \in C(W)$,

$$\rho(g) \cdot (s_W, s_{\partial W}) := (g \cdot s_W, g|_{\partial W} \cdot s_{\partial W})$$

On the other hand, the Kasparov cycle $[[F_{B_1}]]$ is given by

$$(\Gamma(\partial W; F_{B_1}), \varphi, 0)$$

where φ is the representation of $C(\partial W)$ via pointwise multiplication. The Kasparov products $[r_W] \otimes [[F_{B_1}]]$ and $[(E_{B_2}, F_{B_1}, \alpha)] \otimes [\gamma]$ can be explicitly computed. The methods used in the proof of Lemma 3.4.4 in [15] can be used to prove the equality of these KK -elements; the details are left to the reader.

Finally, that $[i!] \otimes [ev_{\mathbb{R}^{2k}}] = [\gamma] \otimes [i|_{\partial W}!]$ follows from the commutative diagram considered in Remark 3.3.

These computations imply that

$$\begin{aligned} [f|_{\partial W}] \otimes [[\xi_{B_1}]] \otimes [i|_{\partial W}!] &= [f] \otimes [r_W] \otimes [[\xi_{B_1}]] \otimes [i|_{\partial W}!] \\ &= [f] \otimes [[\xi]] \otimes [\gamma] \otimes [i|_{\partial W}!] \\ &= [f] \otimes [[\xi]] \otimes [i!] \otimes [ev_{\mathbb{R}^{2k}}] \end{aligned}$$

This completes the proof that the diagram given at the beginning of the proof commutes. The Five Lemma and the fact that μ_{B_1} and μ_{B_2} are isomorphisms for X a finite CW-complex (see for example [19]) then imply that μ is also an isomorphism. \square

Definition 3.11. Let (W, ξ, f) be a cycle in $K_p(X; \phi)$ ($p = 0$ or 1). Then, for k sufficiently large, there exists, $i : W \rightarrow H^{2k+p}$, a K -oriented neat embedding of W into the halfspace H^{2k+p} . The topological index of (W, ξ, f) is defined to be

$$\text{ind}_{\text{top}}(W, \xi, f) := i!(\xi)$$

Using Bott periodicity, we can (and will) consider this as an element in $K_p(SC_\phi)$.

Corollary 3.12. *The topological index map is well-defined (as a map from $K_p(X; \phi)$ to $K_p(H^2, \mathbb{R}; \phi)$). Moreover, in the case when X is a point, the topological index map is an isomorphism.*

Proof. The first statement follows from the fact that the topological index map is given by the composition, $c^* \circ \mu$, where $c : \mathbb{C} \rightarrow C(X)$ is the natural inclusion and μ is the isomorphism in Theorem 3.10. The second statement follows as a special case of Theorem 3.10. \square

Remark 3.13. Note that this result is equivalent to the following commutative diagram:

$$\begin{array}{ccc}
 K_*(X; \phi) & \xrightarrow{\mu} & KK^*(C(X), SC_\phi) \\
 \searrow \text{ind}_{top} & & \swarrow c^* \\
 & K_*(SC_\phi) &
 \end{array}$$

This diagram does not encode an index theorem. The relevant index theorem requires first the definition of an analytic index map. We will take this up in the next section; the index theorem is Theorem 4.7.

4. AN INDEX MAP VIA BOUNDARY CONDITIONS

Our goal is the construction of an analytic index map from $K_0(X; \phi)$ to $K_1(C_\phi)$. This index map will be defined under the assumption that $K_1(B_1) \cong 0$. We make use of higher Atiyah-Patodi-Singer index theory. In the next subsection, we discuss the relationship between the higher Atiyah-Patodi-Singer index and vector bundle modification. This discussion is written in a self-contained manner as its main result is of some independent interest; it will also serve as an introduction to higher Atiyah-Patodi-Singer index theory and the notation required for the second subsection. The reader is directed to [12] and [18] and references therein for further details on this theory.

4.1. Higher Atiyah-Patodi-Singer index theory and vector bundle modification. Following [18], we introduce some notation. Let W be a connected, compact, Riemannian $spin^c$ -manifold with boundary with a product structure in a neighborhood of the boundary. Also let W_{cyl} denote the manifold obtained from W by attaching a cylindrical end to the boundary of W . In other words, there exists $\epsilon > 0$, submanifold $Z_\epsilon \subseteq W_{cyl}$, and $spin^c$ -preserving isometry $e : Z_\epsilon \rightarrow (-\epsilon, \infty) \times \partial W$ such that

$$W = W_{cyl} - e^{-1}((0, \infty) \times \partial W)$$

We also let $Z := \mathbb{R} \times \partial W$, $U_\epsilon := e^{-1}((-\epsilon, 0]) \subseteq W$, and p denote the projection $U_\epsilon \rightarrow \partial W$. In an abuse of notation, we refer to $\partial W \times (0, \infty)$ when working with $e^{-1}((0, \infty) \times \partial W)$.

Let B be a unital C^* -algebra, E_B be a finitely generated projective B -Hilbert module bundle over W and S_W be the spinor bundle associated with the $spin^c$ -structure on W . Then, $\mathcal{E} := S_W \otimes_{\mathbb{C}} E_B$ has a natural Dirac B -bundle structure in the sense of [18, Section 2]. We denote the Clifford connection on this bundle by ∇ and assume that this construction respects the product structure of $\partial W \subseteq W$. In particular, $\mathcal{E}|_{U_\epsilon} = p^*(\mathcal{E}|_{\partial W})$.

Let $\not{D}_{\partial W}$ denote the Dirac operator associated to the Dirac bundle restricted to the boundary of W . In [18] (also see [12]), a number of operators are associated to the data introduced in the previous two paragraphs. First, however, we must perturb the operator on the boundary. Let A be a selfjoint operator in $\mathcal{B}(L^2(\partial W; S_{\partial W} \otimes (E_B|_{\partial W})))$ such that $\not{D}_{\partial W} + A$ is invertible. The existence of A

follows from the vanishing of the index of $\not{D}_{\partial W}$ (see [12] for further details). In fact, we can assume that A is a smoothing operator. Following the notation of [18], let $D_W(A)$ be the operator on W associated to higher Atiyah-Patodi-Singer boundary conditions and $D_{W_{cyl}}(A)$ be the Dirac operator on W_{cyl} perturbed on the cylinder by A . A detailed discussion of these operators (in particular, their construction) can be found in [18, Section 2].

Since the latter operator is of more importance in this work, we only give the details of its construction. Let \not{D} denote the Dirac operator associated to the Dirac bundle \mathcal{E} over W_{cyl} and $\chi : W \rightarrow [0, 1]$ be a function which satisfies

- (1) $\text{supp}(f) \subseteq \partial W \times (-\frac{3\epsilon}{4}, \infty)$;
- (2) For each $w \in \partial W \times (0, \infty)$, $f(w) = 1$;

Denote the Clifford action by c and the vector in the normal direction by x_1 . Then $D_{W_{cyl}}(A)$ is defined to be the closure of the operator

$$\not{D} - c(dx_1)\chi A$$

It has an associated index in the K -theory of B . The reader can find further details on this construction in [18].

Our goal is to consider vector bundle modification as it relates to higher index theory for manifolds with boundary. As such, let V be a $spin^c$ -vector bundle over W with even-dimensional fibers. Further, assume that V respects the product structure of $\partial W \subseteq W$. Using the vector bundle modification operation, we obtain from W and V a $spin^c$ -manifold $\hat{W} := S(V \oplus \mathbf{1})$ where $\mathbf{1}$ denotes the trivial real line bundle over W ; note that \hat{W} is a fiber bundle over W . Moreover, since W is connected, the fiber is S^{2k} for some $k \in \mathbb{N}$. By extending the vector bundle V to W_{cyl} , we can also consider the vector bundle modification of W_{cyl} . We denote the resulting manifold by \hat{W}_{cyl} .

The vector bundle modification operation affects the bundle data on W as follows. Let β denote the Bott bundle over \hat{W} ; it is a vector bundle and its construction can be found in [1]. Then the Hilbert B -bundle on \hat{W} is given by $\pi^*(E_B) \otimes_{\mathbb{C}} \beta$ where $\pi : \hat{W} \rightarrow W$ is the projection map. By the two out of three property of $spin^c$ -vector bundles (see for example [3]), there is a $spin^c$ -structure on \hat{W} . We let $S_{\hat{W}}$ denote the spinor bundle associated $spin^c$ -structure and $\hat{\mathcal{E}}$ denote the B -Dirac bundle $S_{\hat{W}} \otimes \pi^*(E_B) \otimes_{\mathbb{C}} \beta$. These constructions can also be applied to \hat{W}_{cyl} . In an abuse of notation, we denote the Bott bundle over \hat{W}_{cyl} also by β and the B -Dirac bundle over \hat{W}_{cyl} also by $\hat{\mathcal{E}}$. Based on this discussion, we can construct the associated operators discussed in the preceeding paragraphs (this time on the manifolds \hat{W} and \hat{W}_{cyl}). However, the construction of these operators involved the choice of operator A . We would like to construct from a choice of A on the base W a natural choice of such an operator for \hat{W} .

The desired construction and the main result of this subsection are the content of the next proposition. The proof requires the following lemma which is a well-known result in KK -theory (cf. [3, Lemma 2.7] in the case of analytic K -homology).

Lemma 4.1. *Let (\mathcal{E}, ρ, F) be a Kasparov cycle representing a class in $KK^0(A, B)$ and suppose that $T \in \mathcal{L}(\mathcal{E})$ is a self-adjoint, odd-graded involution which commutes with action of A and anticommutes with F . Then the class (in $KK^0(A, B)$) of (\mathcal{E}, ρ, F) is zero.*

Proposition 4.2. *We use the notation introduced in the previous few paragraphs. For example, W denotes a compact spin^c -manifold with boundary, E_B a bundle of finitely generate projective Hilbert B -modules, and V a spin^c -vector bundle over W with even dimensional fibers. Then, given a choice of Dirac operator on W and selfadjoint operator A (see above), there exists a Dirac operator on \hat{W} and selfadjoint operator \hat{A} such that*

$$\text{ind}_{APS}(D_W(A)) = \text{ind}_{APS}(D_{\hat{W}}(\hat{A})) \in K_*(B)$$

The reader should note that the Dirac operator on \hat{W} and \hat{A} are defined in the proof.

Remark 4.3. A word or two on the statement of the proposition seems in order. Perhaps most importantly, the proposition does not imply that the higher Atiyah-Patodi-Singer index is invariant under vector bundle modification. The specific choice of spectral section and Dirac operator on the manifold \hat{W} are important to the proof. These operators are constructed via a partition of unity argument.

In this regard, the statement of the theorem is unsatisfying in a number ways. In particular, one would hope to find a *canonical* construction of a spectral section on the modified manifold given one on the base; our construct of the spectral section is quite ad hoc. Despite this, the theorem suffices for our purposes.

Proof. The structure of the proof is as follows. By [18, Propostion 2.1], the proof will be complete upon showing that the operators $D_{W_{\text{cyl}}}(A)$ and $D_{\hat{W}_{\text{cyl}}}(\hat{A})$ have the same index; of course, the construction of \hat{A} and the Dirac operator are also required. Apart from these constructions, the proof consists of two steps

- (1) proving the result in the case when V is a trivial vector bundle. The reader should note that in this case, $\hat{W} = W \times S^{2k}$;
- (2) using a partition unity argument to treat the case of general V ;

As such, the steps in the proof are the same as those in the proof of Proposition 3.6 in [3]. The case when W is even dimensional is considered in detail; the odd case is left to the reader.

The case when V is a trivial bundle is considered first. In which case $\hat{W} = W \times S^{2k}$ and we can take the product of the Dirac operators to form the Dirac operator on $W \times S^{2k}$. Let $\hat{A} := \Psi(A \otimes I)\Psi^{-1}$ where

$$\Psi : ((S_W \otimes_{\mathbb{C}} E_A)|_{\partial W}) \boxtimes \beta \rightarrow S_{\partial W} \otimes_{\mathbb{C}} E_A|_{\partial W} \otimes_{\mathbb{C}} \beta$$

is defined as in [18, page 6]. It follows that \hat{A} is selfadjoint and $\not{D}_{\partial W \times S^{2k}} + \hat{A}$ is invertible. To see that the latter of these statements holds, one notes that

$$(\not{D}_{\partial W \times S^{2k}} + \hat{A})^2 = (\not{D}_{\partial W} + A)^2 \otimes I + I \otimes \not{D}_{S^{2k}}^2$$

where $\not{D}_{\partial W}$ and $\not{D}_{S^{2k}}$ are respectively the Dirac operators on ∂W and S^{2k} . That this operator is invertible follows since $(\not{D}_{\partial W} + A)^2$ is invertible and both operators are positive. The invertibility of the original operator follows since it is selfadjoint; in particular,

$$(\not{D}_{\partial W} + A)^2 = (\not{D}_{\partial W} + A)^*(\not{D}_{\partial W} + A)$$

Let $\hat{\otimes}$ denote the (graded) algebraic tensor product and \mathcal{S} denote the spinor bundle of S^{2k} . Then, on

$$C_c^\infty(M_{\text{cyl}}; \mathcal{E}) \hat{\otimes} C^\infty(S^{2k}; \mathcal{S} \otimes \beta) \subset C_c^\infty(M_{\text{cyl}} \times S^{2k}; \hat{\mathcal{E}})$$

the twisted Dirac operator on $W_{cyl} \times S^{2k}$ has the form

$$\not{D}_{W_{cyl}} \hat{\otimes} I + I \hat{\otimes} \not{D}_{S^{2k}}$$

In fact, operator $\not{D}_{W_{cyl} \times S^{2k}} - c(dx_1)\hat{\chi}\hat{A}$ also decomposes in this way. That is, on $C_c^\infty(M_{cyl}; \mathcal{E}) \hat{\otimes} C^\infty(S^{2k}; \mathcal{S} \otimes \beta)$, it is equal to

$$(\not{D}_{W_{cyl}} + c(dx_1)\chi A) \hat{\otimes} I + I \hat{\otimes} \not{D}_{S^{2k}}$$

Here, the reader should note that $\hat{\chi}$ and χ are related as follows: $\hat{\chi} : W_{cyl} \times S^{2k} \rightarrow [0, 1]$ is defined via $\hat{\chi}(w, z) := \chi(w)$. The closure of the above operator (i.e., $D_{W_{cyl} \times S^{2k}}(\hat{A})$) therefore has the form

$$D_{W_{cyl} \times S^{2k}}(\hat{A}) = D_{W_{cyl}}(A) \hat{\otimes} I + I \hat{\otimes} D_{S^{2k}}$$

as an operator on

$$L^2(W_{cyl} \times S^{2k}; \hat{\mathcal{E}}) \cong L^2(W_{cyl}; \mathcal{E}) \hat{\otimes} L^2(S^{2k}; \mathcal{S} \otimes \beta)$$

We now apply techniques from [3]. Namely, the Hilbert module on which the operator $D_{W_{cyl} \times S^{2k}}(\hat{A})$ acts (as an unbounded operator) decomposes as follows

$$\begin{aligned} L^2(W_{cyl} \times S^{2k}; \hat{\mathcal{E}}) &\cong L^2(W_{cyl}; \mathcal{E}) \hat{\otimes} L^2(S^{2k}; \mathcal{S} \otimes \beta) \\ &\cong (L^2(W_{cyl}; \mathcal{E}) \hat{\otimes} \ker(D_{S^{2k}})) \oplus (L^2(W_{cyl}; \mathcal{E}) \hat{\otimes} \ker(D_{S^{2k}})^\perp) \end{aligned}$$

Moreover, the operator respects this decomposition. That is, if P denotes the projection onto $L^2(W_{cyl}; \mathcal{E}) \hat{\otimes} \ker(D_{S^{2k}})$, then

$$D_{W_{cyl} \times S^{2k}}(\hat{A}) = PD_{W_{cyl} \times S^{2k}}(\hat{A})P + P^\perp D_{W_{cyl} \times S^{2k}}(\hat{A})P^\perp$$

The operator $PD_{W_{cyl} \times S^{2k}}(\hat{A})P$ acts as $D_{W_{cyl}}(A)$ on $L^2(W_{cyl}; \mathcal{E}) \hat{\otimes} \ker(D_{S^{2k}})$; to see this, the reader should note that $\ker(D_{S^{2k}})$ is one dimensional and is given by the span of an even section (see [3, Proposition 3.11]).

This reduces the proof (of the special case when V is trivial) to showing that $\text{ind}(P^\perp D_{W_{cyl} \times S^{2k}}(\hat{A})P^\perp) = 0$. To this end, consider the operator $\gamma \otimes T$ where γ is the grading operator and T is the partial isometry in the polar decomposition of $D_{S^{2k}}$. As the reader can verify (see also [3, Section 4]) this operator is an odd graded involution on $L^2(M; \mathcal{E}) \hat{\otimes} \ker(D_{S^{2k}})^\perp$. Moreover, $\gamma \otimes T$ anti-commutes with $P^\perp D_{W_{cyl} \times S^{2k}}^{prod}(\hat{A})P^\perp$. Lemma 4.1 implies that $\text{ind}(P^\perp D_{M \times S^{2k}} P^\perp)$ is zero. This completes the proof in the case when V is a trivial vector bundle.

The general case is now considered. As such, let V be a general $spin^c$ -vector bundle with even-dimensional fibers. We must construct the Dirac operator and the operator, \hat{A} .

But, we begin with the Dirac operator on the boundary of \hat{W} . Again, the reader should compare our construction here with the one in the proof of Proposition 3.6 in [3]. Denote the principal $spin^c(2k)$ -bundle associated to the $spin^c$ -structure of $\partial\hat{W}$ by $\mathcal{P}_{\partial\hat{W}}$. Let $\{U_i\}$ be a finite open cover of $\partial\hat{W}$ such that $(\mathcal{P}_{\partial\hat{W}})|_{U_i}$ is trivial for each i . For each i , fix a particular trivialization. Also, fix a smooth partition of unity $\{\sigma_i\}$ which is subordinate to the cover. For each U_i , we define an operator \not{D}_i by letting it act as the Dirac operator of the boundary in the U_i -direction and the identity in the direction of the group. Finally, let R equal the operator obtained by averaging $\sum_i \sigma_i \not{D}_i \sigma_i$ over the action of the group. The operator, R , is a first-order,

formally self-adjoint, equivariant operator (see [3]). Moreover, we have that the Dirac operator on $\partial\hat{W}$ has the form

$$\hat{\phi}_{\partial\hat{W}} = R \hat{\otimes} I + I \hat{\otimes} \hat{\phi}_{S^{2k}}$$

The \hat{A} operator is constructed in a similar way. For each i , let A_i be the operator which acts as A in the U_i direction and the identity in the group direction. Let \tilde{A} be the operator obtained by averaging $\sum_i \sigma_i A_i \sigma_i$ over the action of the group. Finally, let $\hat{A} = \tilde{A} \hat{\otimes} I$. Then, as in the case of modification by a trivial vector bundle, we have that

- (1) \hat{A} is a self-adjoint operator;
- (2) $\hat{\phi}_{\partial\hat{W}} + \hat{A}$ is invertible;

One should also note that

$$\hat{\phi}_{\partial\hat{W}} + \hat{A} = (R + \tilde{A}) \hat{\otimes} I + I \hat{\otimes} D_{S^{2k}}$$

This completes the construction of \hat{A} .

The construction of the operator on \hat{W}_{cyl} proceeds as follows. Let \mathcal{P}_{cyl} denote the principal $spin^c(2k)$ -bundle associated with the $spin^c$ -structure of \hat{W}_{cyl} . Let $\{V_i\}$ be a finite open cover of \hat{W}_{cyl} such that,

- (1) the bundle, $\mathcal{P}_{cyl}|_{V_i}$ is trivial for each i ;
- (2) for each i , $V_i \cap \partial(\hat{W} \times (-\epsilon, \infty))$ is empty or equal to $U_j \times (-\epsilon, \infty)$ for some j ;

The reader should note that although \hat{W}_{cyl} is not compact such a cover exists.

We will use this cover to construct the Dirac operator. The reader should recall (see also [3]) that this operator acts on

$$[L^2(\mathcal{P}_{cyl}) \otimes L^2(S^{2k})]^{spin^c(2k)}$$

where we have suppressed the relevant bundles from the notation. In the same way as in the construction of the Dirac operator on the boundary, we can construct an operator, \tilde{R} , such that the Dirac operator takes the form

$$\hat{\phi}_{\hat{W}} = \tilde{R} \hat{\otimes} I + I \hat{\otimes} \hat{\phi}_{S^{2k}}$$

As in the construction of the Dirac operator on the boundary, \tilde{R} is a first-order, formally self-adjoint, equivariant operator. Moreover, the closure of the operator $\hat{\phi}_{\hat{W}} - c(dx_1)\chi\hat{A}$ (i.e., $D_{\hat{W}}(\hat{A})$) decomposes in a similar way. Namely,

$$D_{\hat{W}_{cyl}}(\hat{A}) = \tilde{R}(\hat{A}) \hat{\otimes} I + I \hat{\otimes} D_{S^{2k}}$$

The argument given in the case of a trivial bundle applies here also. It implies that $\text{ind}(D_{\hat{W}}(\hat{A})) = \text{ind}(D_W(A))$. To see this, one needs to check that the restriction of $D_{\hat{W}}(\hat{A})$ to

$$[L^2(\mathcal{P}_{\hat{W}_{cyl}}) \otimes \ker(D_{S^{2k}})]^{spin^c(2k)}$$

acts as $D_W(A)$ once this subspaces has been identified with $L^2(W; \mathcal{E})$. However, this fact follows by the construction of the operators \tilde{R} and \hat{A} . \square

4.2. The analytic index map. For this development, it is more convenient to work with the cycles of the form given in Definition 2.2 (i.e., cycles containing bundle data). In fact, we need only consider cycles in $K_0(pt; \phi)$ since the general index map will be defined by

$$\text{ind}_{ana} : K_0(X; \phi) \rightarrow K_0(pt; \phi) \rightarrow K_1(C_\phi)$$

where the first map is defined at the level of cycles via $(W, (E_{B_2}, F_{B_1}, \alpha), f) \mapsto (W, (E_{B_2}, F_{B_1}, \alpha))$ and the definition of the second map is the main objective of this section; the second map will also be denoted simply as ind_{ana} . To be precise, the geometric data considered in this section is the following. Let

- (1) W be an even-dimensional compact $spin^c$ -manifold with boundary;
- (2) E_{B_2} be a (finitely generated projective) Hilbert B_2 -module bundle over W ;
- (3) F_{B_1} be a (finitely generated projective) Hilbert B_1 -module bundle over ∂W ;
- (4) $\alpha : F_{B_1} \otimes_\phi B_2 \rightarrow E_{B_2}$ is an isomorphism;

The starting point for defining this index is the vanishing of index of the boundary operator (see for example [11]). As such, to define the analytic index map from the $K_0(X; \phi)$ to $K_1(C_\phi)$, we assume that

$$K_1(B_1) \cong 0$$

However, to define it, additional geometric data must be fixed. Let

- (1) g denote a Riemannian metric on W which is a product metric in a neighborhood of ∂W ;
- (2) $\nabla_{F_{B_1}}$ a connection compatible with $g|_{\partial W}$;
- (3) $\nabla_{E_{B_2}}$ a connection which is compatible with g , $\nabla_{F_{B_1}}$, and the bundle isomorphism α ;
- (4) P a spectral section for the operator on the boundary (i.e., $D_{\partial W, F_{B_1}}$);

With all this data fixed, results from [11] imply that there is a well-defined index

$$\text{ind}_{APS}(D_{W, E_{B_2}}^P) \in K_0(B_2)$$

However (as an element of $K_0(B_2)$) it depends on these choices (e.g., the metric, connections, and spectral section). We will show however that the image of this class under $r_* : K_0(B_2) \rightarrow K_1(C_\phi)$ is independent of these choices.

To do so, a number of properties of the higher Atiyah-Patodi-Singer index are required. These properties are that the higher Atiyah-Patodi-Singer index, spectral flow, and difference construction of spectral sections are each functorial. The functorial properties of this index are discussed in [14, Appendix C] while for spectral flow and the difference construction the reader can see [17].

To state these properties precisely, additional notation is required. Recall that $\phi : B_1 \rightarrow B_2$ is a unital $*$ -homomorphism and W is a compact $spin^c$ -manifold with boundary. Further assume that F_{B_1} is a (finitely generated projective) Hilbert B_1 -module bundle over all of W . Let P and Q be spectral sections for $D_{\partial W, E_{B_1}|_{\partial W}}$. The following three properties will be used

$$\begin{aligned} (5) \quad \phi_*(\text{ind}_{APS}^{B_1}(D_{W, E_{B_1}}^P)) &= \text{ind}_{APS}^{B_2}(D_{W, E_{B_1} \otimes_\phi B_2}^{\phi_*(P)}) \\ (6) \quad \phi_*(\text{sf}(D_{\partial W, E_{B_1}|_{\partial W}}, t; P, Q)) &= \text{sf}(D_{\partial W, E_{B_1}|_{\partial W} \otimes_\phi B_2}, t; \phi_*(P), \phi_*(Q)) \\ (7) \quad \phi_*([P - Q]) &= [\phi_*(P) - \phi_*(Q)] \end{aligned}$$

where

- (1) $D_{M,E}^P$ denotes the Dirac operator on M twisted by E with the boundary conditions associated to the spectral section P ;
- (2) ind_{APS} denotes the higher Atiyah-Patodi-Singer index;
- (3) $\text{sf}(\cdot)$ denotes spectral flow (see [17] for further details);
- (4) $[P - Q] \in K_0(B_1)$ denotes the difference class of P and Q (again further details can be found in [12] or [17]);

Definition 4.4. Let $(W, (E_{B_2}, F_{B_1}, \alpha), f)$ be a cycle in $K_0(X; \phi)$ such that

$$\text{ind}_{AS}(D_{\partial W, F_{B_1}}) = 0 \in K_1(B_1)$$

Then,

$$\text{ind}_{ana}(W, (E_{B_2}, F_{B_1}, \alpha), f) := r_*(\text{ind}_{APS}(D_{W, E_{B_2}}^P)) \in K_1(C_\phi)$$

where P is any spectral section for $D_{\partial W, F_{B_1}}$ and $r_* : K_0(B_2) \rightarrow K_1(C_\phi)$ is the map on K -theory induced from the $*$ -homomorphism $r : SB_2 \rightarrow C_\phi$.

Proposition 4.5. Let $(W, (E_{B_2}, F_{B_1}, \alpha), f)$ be a cycle in $K_0(X; \phi)$ and assume that $\text{ind}_{AS}^{B_1}(D_{\partial W}) = 0$. Then, the map

$$(W, (E_{B_2}, F_{B_1}, \alpha), f) \mapsto r_*(\text{ind}_{APS}^{B_2}(D_{W, E_{B_2}}))$$

is well-defined as map on (isomorphism classes of) cycles.

Proof. A proof that the index map is well-defined at the level of cycles amounts to showing the right-hand side of the equation is independent of the choice of metric, connection, and spectral section used to define the higher Atiyah-Patodi-Singer index. We begin with a special case; let

- (1) $\{g_t\}_{t \in [0,1]}$ be a one parameter family of Riemannian metrics on W ;
- (2) $\nabla_{F_{B_1}, t}$ be a one parameter family of connections on F_{B_1} which is compatible with $g_t|_{\partial W}$;
- (3) $\nabla_{E_{B_2}, t}$ be a one parameter family of connections on E_{B_2} which is compatible with g_t and with the family of connections $\nabla_{F_{B_1}, t}$;
- (4) \hat{P}_t be a one parameter family of spectral sections for $D_{\partial W, F_{B_1}}$.

Set $P = \phi_*(\hat{P}_t)$. By functorial properties of spectral sections and the fact that $E_{B_2}|_{\partial W} \cong F_{B_1} \otimes_\phi B_2$, both P_0 and P_1 are spectral sections for $D_{\partial W, E_{B_1}|_{\partial W}}$. Using this data, the following indices are well-defined:

$$\text{ind}_{APS}(D_{W, E_{B_2}}^{P_0}) \text{ and } \text{ind}_{APS}(D_{W, E_{B_2}}^{P_1})$$

Then, [11, Proposition 8] implies that

$$\text{ind}_{APS}(D_{W, E_{B_2}}^{P_0}) - \text{ind}_{APS}(D_{W, E_{B_2}}^{P_1}) = \text{sf}(\{D_{\partial W, (E_{B_2})|_{\partial W}, t}; P_0, P_1\} \in K_0(B_2)$$

where $\text{sf}(D_{\partial W, E|_{\partial W}, t}; P_0, P_1)$ is the spectral flow of the family of operators on the boundary (again see [11]). Functorial properties of spectral flow (i.e., Equation 6) imply that $\text{sf}(D_{\partial W, E_{B_2}, t}; P_0, P_1)$ is in the image of ϕ_* . Exactness (i.e., $r_* \circ \phi_*$) leads to

$$r_*(\text{ind}_{APS}(D_{W, E_{B_2}}^{P_0})) - r_*(\text{ind}_{APS}(D_{W, E_{B_2}}^{P_1})) = 0 \in K_1(C_\phi)$$

This completes the proof of the special case.

The only different for general case is that we cannot assume that the spectral sections, \hat{P}_0 and \hat{P}_1 , are joined via a one-parameter family. However, there does

exists a family of spectral section \hat{Q}_t . As above, set $P_0 = \phi_*(\hat{P}_0)$, $P_1 = \phi_*(\hat{P}_1)$, and $Q_t = \phi_*(\hat{Q}_t)$. Then, using [11, Proposition 8 and Theorem 8], we have

$$\begin{aligned} \text{ind}_{APS}(D_{W,E_{B_2}}^{P_0}) - \text{ind}_{APS}(D_{W,E_{B_2}}^{P_1}) &= \text{ind}_{APS}(D_{W,E_{B_2}}^{P_0}) - \text{ind}_{APS}(D_{W,E_{B_2}}^{P_1}) \\ &\quad - \text{ind}_{APS}(D_{W,E_{B_2}}^{Q_0}) + \text{ind}_{APS}(D_{W,E_{B_2}}^{Q_1}) \\ &\quad + \text{sf}(\{D_{\partial W, (E_{B_2})|_{\partial W}, t}; Q_0, Q_1\}) \\ &= [Q_0 - P_0] + [P_1 - Q_0] \\ &\quad + \text{sf}(\{D_{\partial W, (E_{B_2})|_{\partial W}, t}; Q_0, Q_1\}) \end{aligned}$$

Applying r_* to this equation and using the functorial properties of the difference classes and spectral flow leads to

$$\begin{aligned} r_*(\text{ind}_{APS}(D_{W,E_{B_2}}^{P_0})) - r_*(\text{ind}_{APS}(D_{W,E_{B_2}}^{P_1})) &= (r_* \circ \phi_*)([\hat{Q}_0 - \hat{P}_0] + [\hat{P}_1 - \hat{Q}_0]) \\ &\quad + \text{sf}(\{D_{\partial W, (F_{B_1}), t}; \hat{Q}_0, \hat{Q}_1\}) \end{aligned}$$

Exactness then implies the result. \square

Theorem 4.6. *If $K_1(B_1) \cong 0$, then there is a well-defined index map*

$$\mu_{ana} : K_0(X; \phi) \rightarrow K_1(C_\phi)$$

given by the analytic index map defined in Definition 4.4 (i.e., via the higher Atiyah-Patodi-Singer boundary condition problem on the manifold with boundary in a cycle).

Proof. The assumption $K_1(B_1) \cong 0$ implies that conditions of Proposition 4.5 are satisfied for any cycle in $K_0(X; \phi)$. Thus the index map is well-defined at the level of cycles. We need to show that the map respects the three relations.

Disjoint union/direct sum: This follows from basic properties of the higher Atiyah-Patodi-Singer index.

Bordism: The setup is as follows. Let $(Z, W, (E'_{B_2}, F'_{B_1}, \alpha'), g)$ be a bordism and $(W, (E_{B_2}, F_{B_1}, \alpha), f)$ denote its boundary. Denote by (M, V_{B_1}, h) the $K_*(X; B_1)$ -bordism obtained by restricting the given data to the $spin^c$ manifold with boundary, $\partial W - \text{int}(W)$. Let P and Q be spectral sections for $D_{\partial W, F_{B_1}}$ and $D_{\partial M, V_{B_1}|_{\partial M}}$ respectively and \tilde{P} and \tilde{Q} denote the spectral sections (for $D_{\partial W, F_{B_1} \otimes_\phi B_2}$ and $D_{\partial M, V_{B_1}|_{\partial M} \otimes_\phi B_2}$ respectively) obtained via the $*$ -homomorphism ϕ . Using [11, Theorem 8] and the functorial properties listed above, the indices on the various manifolds (we suppress the bundle data from the notation) involved are related via

$$\begin{aligned} \text{ind}_{APS}^{B_2}(D_W^{\tilde{P}}) + \phi_*(\text{ind}_{APS}^{B_1}(D_M^{I-Q})) &= \text{ind}_{APS}^{B_2}(D_W^{\tilde{P}}) + \text{ind}_{APS}^{B_2}(D_M^{I-\tilde{Q}}) \\ &= \text{ind}_{AS}^{B_2}(D_{W \cup M}) + [\tilde{P} - \tilde{Q}] \\ &= \text{ind}_{AS}^{B_2}(D_{W \cup M}) + \phi_*([P - Q]) \end{aligned}$$

The fact that $r_* \circ \phi_* = 0$ implies that

$$\mu_{ana}(W, (E_{B_2}, F_{B_1}, \alpha), f) = r_*(\text{ind}_{APS}^{B_2}(D_W)) = r_*(\text{ind}_{AS}^{B_2}(D_{W \cup M}))$$

Finally, the bordism invariance of the Mishchenko-Fomenko index and the fact that $W \cup M = \partial Z$ (the bundles respect this bordism) imply that the right-hand side of this equation vanishes. This proves the required bordism invariance.

Vector bundle modification: Let $(W, (E_{B_2}, F_{B_1}, \alpha), f)$ denote a cycle and V a

$spin^c$ -vector bundle of even rank over W . Since the higher Atiyah-Patodi-Singer index respects disjoint union, we may assume that W is connected. Using Proposition 4.2, we have that

$$\text{ind}_{APS}(D_W(A)) = \text{ind}_{APS}(D_{\hat{W}}(\hat{A})) \in K_0(B_2)$$

where we have used the notation of Proposition 4.2. However, the definition of μ_{ana} is independent of the choice of spectral section (see Proposition 4.5). As such,

$$\begin{aligned} \mu_{ana}(W, (E_{B_2}, F_{B_1}, \alpha, f)) &= r_*(\text{ind}_{APS}^{B_2}(D_W(A))) \\ &= r_*(\text{ind}_{APS}^{B_2}(D_{\hat{W}}(\hat{A}))) \\ &= \mu_{ana}(W, (E_{B_2}, F_{B_1}, \alpha, f)^V) \end{aligned}$$

□

Theorem 4.7. *Suppose that $K_1(B_1) \cong 0$ so that analytic index is well-defined. Then the topological index and analytic index are equal. In particular, the analytic index in the case when X is a point is an isomorphism.*

Proof. The second statement in the theorem follows from the first and the fact that the topological index is an isomorphism in the case of a point. To prove the first statement, note that both the topological index and analytic index factor through the map

$$K_0(X; \phi) \rightarrow K_0(pt; \phi)$$

defined at the level of cycles via $(W, \xi, f) \mapsto (W, \xi)$. Thus, we need only show that they give the same isomorphism from $K_0(pt; \phi)$ to $K_1(C_\phi)$. Since $K_1(B_1) \cong 0$, the map $r_* : K_0(pt; B_2) \rightarrow K_0(pt; \phi)$ is onto. This implies that given a cycle $(W, \xi) \in K_0(pt; \phi)$ there exists *closed* compact $spin^c$ -manifold M and $\eta \in K^0(M; B_2)$ such that $r(M, \eta) \sim (W, \xi)$. The result follows, since both the topological and analytic index of (W, ξ) are equal to $r_* \circ \text{ind}_{K_0(B_2)}(M, \eta)$ where $\text{ind}_{K_0(B_2)}(M, \eta)$ denotes the Mishchenko-Fomenko index and $r_* : K_*(B_2) \rightarrow K_{*+1}(C_\phi)$ the map on K -theory induced from the natural $*$ -homomorphism $r : SB_2 \rightarrow C_\phi$. □

Remark 4.8. Assuming that $K_1(B_1) \cong 0$, the proof of the previous theorem implies that any index map $K_0(X; \phi) \rightarrow KK(\mathbb{C}, SC_\phi)$ which agrees with the Mishchenko-Fomenko index on cycles without boundary is equal to the topological index map. In particular, this statement holds (up to a factor of -1) for the index map discussed in [5] for the special case when ϕ is the unital inclusion of the complex number into a II_1 -factor. Note that since the index map discussed in [5] takes values in \mathbb{R}/\mathbb{Z} , we must fix the isomorphism from $KK(\mathbb{C}, SC_\phi)$ to \mathbb{R}/\mathbb{Z} to be the one compatible with isomorphism from $KK(\mathbb{C}, N)$ to \mathbb{R} defined via the trace of the II_1 -factor, N .

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